Semilocal categories, local functors and applications

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If R is commutative,
R semilocal $\Leftrightarrow$ R has finitely many maximal ideals.

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- Endomorphism rings of modules of finite Goldie dimension and finite dual Goldie dimension are semilocal rings. (Herbera and Shamsuddin)

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Semilocal Rings and Local Morphisms

Theorem

(**Camps and Dicks**) *A ring R is semilocal if and only if there exists a local morphism $R \to S$ for some semilocal ring S, if and only if there exists a local morphism $R \to S$ for some semisimple artinian ring S.*

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More precisely:

Rings of type n

Proposition

Let S be a ring with Jacobson radical $J(S)$ and $n \geq 1$ be an integer. The following conditions are equivalent:

(a) *The ring $S/J(S)$ is a direct product of n division rings.*

(b) *n is the smallest of the positive integers m for which there is a local morphism of S into a direct product of m division rings.*

(c) *The ring S has exactly n distinct maximal right ideals, and they are all two-sided ideals in S.*

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A ring is said to be *of type n* if it satisfies the equivalent conditions of the Proposition.

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Cyclically presented modules over local rings are of type ≤ 2.

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A finitely generated square-free semisimple module is a module of finite type.

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If M_R is a noetherian module with $M_R/M_RJ(R)$ a semisimple square-free module, then M_R is a module of finite type.

Let E, E' be injective square-free modules of finite Goldie dimension and let $\varphi\colon E \to E'$ be a module morphism. Then $\ker\varphi$ is a module of finite type.

Local Functors

An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between preadditive categories $\mathcal{A}$ and $\mathcal{B}$ is said to be a *local functor* if, for every morphism $f\colon A \to A'$ in $\mathcal{A}$, $F(f)$ isomorphism in $\mathcal{B}$ implies f isomorphism in $\mathcal{A}$.

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The functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \times \mathrm{soc}$ of $\{\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^2 Z\}$ to $\mathrm{vect}\text{-}\mathbb{Z}/p\mathbb{Z} \times \mathrm{vect}\text{-}\mathbb{Z}/p\mathbb{Z}$ is local but not isomorphism reflecting.

Jacobon radical

Lemma

Let $\mathcal{A}$ be a preadditive category and A, B objects of $\mathcal{A}$. The following conditions are equivalent for a morphism $f: A \to B$:
(a) $1_A - gf$ *has a left inverse for every morphism* $g: B \to A$;
(b) $1_B - fg$ *has a left inverse for every morphism* $g: B \to A$;
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Let $\mathcal{J}(A, B)$ be the set of all morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ satisfying the equivalent conditions of the Lemma. Then $\mathcal{J}$ turns out to be an ideal of the category $\mathcal{A}$, called the *Jacobson radical* of $\mathcal{A}$.

Local Functors and Jacobson radical

- The canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{J}$, with $\mathcal{A}$ a preadditive category and $\mathcal{J}$ its Jacobson radical, is a local functor.

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- More generally, if $\mathcal{A}$ is a preadditive category and $\mathcal{I}$ is any ideal of $\mathcal{A}$ contained in the Jacobson radical, the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ is a local functor.
- Conversely, the kernel of any local functor $F\colon \mathcal{A} \to \mathcal{B}$ is contained in the Jacobson radical of $\mathcal{A}$.
- A full functor $F\colon \mathcal{A} \to \mathcal{B}$ is a local functor if and only if its kernel is contained in the Jacobson radical $\mathcal{J}$ of $\mathcal{A}$.

The canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}_1 \times \cdots \times \mathcal{A}/\mathcal{I}_n$

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Problem: *Let $\mathcal{A}$ be a preadditive category and let $\mathcal{I}_1, \dots, \mathcal{I}_n$ be ideals of $\mathcal{A}$.*

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Problem: *Let $\mathcal{A}$ be a preadditive category and let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be ideals of $\mathcal{A}$.*
When is the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}_1 \times \cdots \times \mathcal{A}/\mathcal{I}_n$ a local functor?

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$$\mathbb{Z}\langle x, y_1\rangle \subset \mathbb{Z}\langle x, y_1, y_2\rangle \subset \mathbb{Z}\langle x, y_1, y_2, y_3\rangle \subset \dots$$

of non-commutative integral domains, where $\mathbb{Z}\langle x, y_1, \dots, y_n\rangle$ indicates the ring of polynomials in the non-commutative indeterminates $x, y_1, \dots, y_n$ with coefficients in $\mathbb{Z}$.

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Proposition

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(c)

$$p_n = \sum_{1\leq i\leq n} y_i - \sum_{1\leq i_1<i_2\leq n} y_{i_1} x y_{i_2} + \\ + \sum_{1\leq i_1<i_2<i_3\leq n} y_{i_1} x y_{i_2} x y_{i_3} - \cdots + (-1)^{n-1} y_1 x y_2 x \ldots x y_n$$

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(c) $\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_n \subseteq \mathcal{J}$*.*

Corollary

Let $\mathcal{I}_1, \dots, \mathcal{I}_n$ *be* n *ideals of a preadditive category* $\mathcal{B}$, *and let* $\mathcal{C}$ *be the full subcategory of* $\mathcal{B}$ *whose objects are all the objects* A *of* $\mathcal{B}$ *with* $\mathcal{I}_1(A, A) \cap \dots \cap \mathcal{I}_n(A, A) \subseteq J(\mathrm{End}_{\mathcal{B}}(A))$.

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Let $\mathcal{I}_1, \dots, \mathcal{I}_n$ be n ideals of a preadditive category $\mathcal{B}$, and let $\mathcal{C}$ be the full subcategory of $\mathcal{B}$ whose objects are all the objects A of $\mathcal{B}$ with $\mathcal{I}_1(A,A) \cap \cdots \cap \mathcal{I}_n(A,A) \subseteq J(\mathrm{End}_{\mathcal{B}}(A))$. Then the ideal $\mathcal{I}_1 \cap \cdots \cap \mathcal{I}_n$ restricted to the full subcategory $\mathcal{C}$, is contained in the Jacobson radical $\mathcal{J}$ of $\mathcal{C}$, so that the canonical functor $C \colon \mathcal{C} \to \mathcal{C}/\mathcal{I}_1 \times \cdots \times \mathcal{C}/\mathcal{I}_n$ is local.

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Semilocal Categories

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- The full subcategory of all modules of finite Goldie dimension and finite dual Goldie dimension.

Local functors and maximal ideals

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Proposition

Let $\mathcal{A}$ *be a preadditive category and* $\mathcal{I}_1, \dots, \mathcal{I}_n$ *be finitely many ideals of* $\mathcal{A}$*.*

(a) *If the canonical functor* $\mathcal{A} \to \mathcal{A}/\mathcal{I}_1 \times \mathcal{A}/\mathcal{I}_2 \times \cdots \times \mathcal{A}/\mathcal{I}_n$ *is a local functor, then every maximal ideal of* $\mathcal{A}$ *contains at least one of the ideals* $\mathcal{I}_i$*.*

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(b) *If the category $\mathcal{A}$ is semilocal and every maximal ideal of $\mathcal{A}$ contains at least one of the ideals $\mathcal{I}_i$, then the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}_1 \times \mathcal{A}/\mathcal{I}_2 \times \cdots \times \mathcal{A}/\mathcal{I}_n$ is local.*

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(b) *If the category $\mathcal{A}$ is semilocal and every maximal ideal of $\mathcal{A}$ contains at least one of the ideals $\mathcal{I}_i$, then the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}_1 \times \mathcal{A}/\mathcal{I}_2 \times \cdots \times \mathcal{A}/\mathcal{I}_n$ is local.*

Proposition

If $\mathcal{C}$ is a semilocal category, the canonical functor $F \colon \mathcal{C} \to \oplus_{\mathcal{M} \in \mathrm{Max}(\mathcal{C})} \mathcal{C}/\mathcal{M}$ is local.

Local functor implies isomorphism reflecting functor for semilocal categories

Theorem

If $\mathcal{A}$ is a semilocal category and $\mathcal{I}_1, \dots, \mathcal{I}_n$ are ideals of $\mathcal{A}$ such that the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}_1 \times \cdots \times \mathcal{A}/\mathcal{I}_n$ is local, then two objects of $\mathcal{A}$ are isomorphic in $\mathcal{A}$ if and only if they are isomorphic in $\mathcal{A}/\mathcal{I}_i$ for every $i = 1, 2, \dots, n$.

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R a ring, ideals in the category $\mathrm{Mod}\text{-}R$.
(1) The ideal Δ, defined by

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Notice that $\Delta + \Sigma$ is not the improper ideal of Mod-R in general. For instance, if R is a division ring, then both Δ and Σ are the zero ideal.

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Theorem

The product functor $\mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}R/\Delta \times \mathrm{Mod}\text{-}R/\Sigma$ *is a local functor.*

Spectral Category (Gabriel and Oberst)

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where the direct limit is taken over the family of all essential subobjects A' of A.

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There is a canonical, left exact, covariant, additive functor $P \colon \mathcal{A} \to \mathrm{Spec}\ \mathcal{A}$, which is the identity on objects and maps any morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ to its canonical image in $\mathrm{Hom}_{\mathrm{Spec}\ \mathcal{A}}(A, B)$.

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The ideal Δ is the kernel of the functor P.

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The category $\mathcal{A}'$:
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$$\mathrm{Hom}_{\mathcal{A}'}(A, B) := \varinjlim \mathrm{Hom}_{\mathcal{A}}(A, B/B'),$$

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Example 2

$\Delta^{(1)} =$ kernel of the right derived functor

$$P^{(1)} \colon \mathrm{Mod}\text{-}R \to \mathrm{Spec}(\mathrm{Mod}\text{-}R)$$

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Theorem

The product functor $\mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}R/\Delta \times \mathrm{Mod}\text{-}R/\Delta^{(1)}$ *is a local functor.*

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Theorem

The product functor $\mathcal{C} \to \mathcal{C}/\Sigma \times \mathcal{C}/\Sigma_{(1)}$ is a local functor.

An application

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Similarly, M and N *belong to the same epigeny class* (written $[M]_e = [N]_e$) if there exist an epimorphism $M \to N$ and an epimorphism $N \to M$.

Weak Krull-Schmidt for uniserial modules

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Let $U_1, \ldots, U_n, V_1, \ldots, V_t$ be non-zero uniserial right modules over an arbitrary ring R. Then $U_1 \oplus \cdots \oplus U_n \cong V_1 \oplus \cdots \oplus V_t$ if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \ldots, n\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \ldots, n$.

Ideal Δ and monogeny classes

If two modules A_R, B_R are isomorphic objects in the category $\mathrm{Mod}\text{-}R/\Delta$, then they have the same monogeny class

The general result

Theorem

(Weak Krull-Schmidt Theorem for additive categories) *Let $\mathcal{A}$ be an additive category and $\mathcal{I}_1, \dots, \mathcal{I}_n$ be ideals of $\mathcal{A}$ such that the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{I}_1 \times \cdots \times \mathcal{A}/\mathcal{I}_n$ is a local functor. Let $A_i,\ i = 1, 2, \dots, t$, and $B_j,\ j = 1, 2, \dots, m$, be objects of $\mathcal{A}$ such that the endomorphism rings $\mathrm{End}_{\mathcal{A}/\mathcal{I}_k}(A_i)$ are local rings for every $i = 1, 2, \dots, t$ and every $k = 1, 2, \dots, n$ and the endomorphism rings $\mathrm{End}_{\mathcal{A}/\mathcal{I}_k}(B_j)$ are all local rings for every $j = 1, 2, \dots, m$ and every $k = 1, 2, \dots, n$. Then $A_1 \oplus \cdots \oplus A_t \cong B_1 \oplus \cdots \oplus B_m$ if and only if $t = m$ and there exist n permutations $\sigma_k,\ k = 1, 2, \dots, n$, of $\{1, 2, \dots, t\}$ with A_i isomorphic to $B_{\sigma_k(i)}$ in $\mathcal{A}/\mathcal{I}_k$ for every $i = 1, 2, \dots, t$ and every $k = 1, 2, \dots, n$.*

A curiosity: Birkhoff's Theorem for skeletally small preadditive categories

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A ring R is *subdirectly irreducible* if the intersection of all non-zero two-sided ideals of R is non-zero.

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Birkhoff's Theorem. *Any ring is a subdirect product of subdirectly irreducible rings.*

Subdirectly irreducible rings

R subdirect product of a family of rings R_i $(i \in I)$ = there is an embedding $R \hookrightarrow \prod_{i \in I} R_i$ in such a way that $\pi_j(R) = R_j$ for each projection $\pi_j \colon \prod_{i \in I} R_i \to R_j$.

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$R \hookrightarrow \prod_{i \in I} R_i$ is called a *subdirect embedding.*

R is subdirectly irreducible if and only if for every family of rings R_i and every subdirect embedding $\varepsilon \colon R \to \prod_{i \in I} R_i$, there exists an index $i \in I$ such that $\pi_i \varepsilon \colon R \to R_i$ is an isomorphism.

Birkhoff's Theorem

Birkhoff's Theorem hold for rings, right modules, lattices, any universal algebra.

Birkhoff's Theorem for skeletally small preadditive categories

Let $\mathcal{A}_i$ $(i \in I)$ be a family of preadditive categories,
$\prod_{i\in I} \mathcal{A}_i$ the product category and,
for every $j \in I$, $P_j \colon \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_j$ be the canonical projection.
We say that a preadditive category $\mathcal{A}$ is a *subdirect product* of the indexed family $\{\, \mathcal{A}_i \mid i \in I \,\}$ of preadditive categories if $\mathcal{A}$ is a subcategory of the product category $\prod_{i\in I} \mathcal{A}_i$ and, for every $i \in I$, the restriction $P_i|_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}_i$ is a full functor that induces an onto mapping $\mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{A}_i)$.

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A functor $F\colon \mathcal{A} \to \mathcal{B}$ between two categories $\mathcal{A}, \mathcal{B}$ is *dense* if every object of $\mathcal{B}$ is isomorphic to $F(A)$ for some object A of $\mathcal{A}$.

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A *subdirect embedding* $F\colon \mathcal{A} \to \prod_{i \in I} \mathcal{A}_i$ is a faithful additive functor F such that, for every $i \in I$, $P_i F\colon \mathcal{A} \to \mathcal{A}_i$ is a dense full functor.

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A preadditive category $\mathcal{A}$ is *subdirectly irreducible* if, for every subdirect embedding $F\colon \mathcal{A} \to \prod_{i\in I} \mathcal{A}_i$, there exists an index $i \in I$ such that $P_iF\colon \mathcal{A} \to \mathcal{A}_i$ is a category equivalence.

Theorem

The following conditions are equivalent for a skeletally small preadditive category $\mathcal{A}$:

(1) *$\mathcal{A}$ is subdirectly irreducible.*

(2) *There exists a nonzero ideal $\mathcal{I}$ of $\mathcal{A}$ such that $\mathcal{I} \subseteq \mathcal{J}$ for every nonzero ideal $\mathcal{J}$ of $\mathcal{A}$.*

(3) *If the intersection of a set $\mathcal{F}$ of ideals of $\mathcal{A}$ is the zero ideal, then one of the ideals in $\mathcal{F}$ is zero.*

(4) *There exist two objects $\overline{A}$ and $\overline{B}$ of $\mathcal{A}$ and a nonzero morphism $\overline{f} \colon \overline{A} \to \overline{B}$ such that, for every nonzero morphism $f \colon A \to B$ in $\mathcal{A}$, there exist a positive integer n and morphisms $g_1, \dots, g_n \colon \overline{A} \to A$ and $h_1, \dots, h_n \colon B \to \overline{B}$ with $\overline{f} = \sum_{i=1}^{n} h_i f g_i$.*

(5) *There exist two objects $\overline{A}$ and $\overline{B}$ of $\mathcal{A}$ with the following two properties:* (a) *The $(\mathrm{End}_{\mathcal{A}}(\overline{B}), \mathrm{End}_{\mathcal{A}}(\overline{A}))$-bimodule $\mathcal{A}(\overline{A}, \overline{B})$ is an essential extension of a simple $(\mathrm{End}_{\mathcal{A}}(\overline{B}), \mathrm{End}_{\mathcal{A}}(\overline{A}))$-subbimodule;* (b) *For every A, B objects of $\mathcal{A}$ and nonzero morphism $f \colon A \to B$ in $\mathcal{A}$, one has that $\mathcal{A}(B, \overline{B}) f \mathcal{A}(\overline{A}, A) \neq 0$.*

Birkhoff's Theorem for skeletally small preadditive categories

For every skeletally small preadditive category $\mathcal{A}$, there exists a subdirect embedding of $\mathcal{A}$ into a direct product of subdirectly irreducible preadditive categories.

An example

Let $\mathcal{A}, \mathcal{A}_f$ be the full subcategories of Ab whose objects are all torsion-free abelian groups and all torsion-free abelian groups of finite rank, respectively.

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Let $\mathcal{A}, \mathcal{A}_f$ be the full subcategories of Ab whose objects are all torsion-free abelian groups and all torsion-free abelian groups of finite rank, respectively. Then $\mathcal{A}$ and $\mathcal{A}_f$ are subdirectly irreducible categories and their least nonzero ideal is generated by the inclusion $\varepsilon\colon \mathbb{Z} \to \mathbb{Q}$.

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The case of $\mathrm{Mod}\text{-}R$

Theorem

Let R be a ring, $\mathcal{S}$ a set of representatives of the simple right R-modules up to isomorphism, and $\mathcal{M}$ the set of all minimal nonzero ideals of $\mathrm{Mod}\text{-}R$. Then:

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(1) *Every nonzero ideal of $\mathrm{Mod}\text{-}R$ contains an element of $\mathcal{M}$.*
(2) *There is a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{M}$. If $S_R \in \mathcal{S}$, the corresponding element $\mathcal{J}_{S_R}$ of $\mathcal{M}$ is the ideal of $\mathrm{Mod}\text{-}R$ generated by any morphism $f \colon R_R \to E(S_R)$ with image S_R.*

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Theorem

Let R be a ring, $\mathcal{S}$ a set of representatives of the simple right R-modules up to isomorphism, and $\mathcal{M}$ the set of all minimal nonzero ideals of $\mathrm{Mod}\text{-}R$. Then:
(1) *Every nonzero ideal of $\mathrm{Mod}\text{-}R$ contains an element of $\mathcal{M}$.*
(2) *There is a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{M}$. If $S_R \in \mathcal{S}$, the corresponding element $\mathcal{J}_{S_R}$ of $\mathcal{M}$ is the ideal of $\mathrm{Mod}\text{-}R$ generated by any morphism $f\colon R_R \to E(S_R)$ with image S_R.*

Corollary

Let R be a ring. The category $\mathrm{Mod}\text{-}R$ is subdirectly irreducible if and only if R has a unique simple right module up to isomorphism.

The torsion theory $(\mathcal{T}_S, \mathcal{F}_S)$

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$\Rightarrow$ $(\mathcal{T}_S, \mathcal{F}_S)$ is hereditary ($=$ $\mathcal{T}_S$ is closed under submodules).

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Let $t_S\colon \mathrm{Mod}\text{-}R \to \mathrm{Mod}\text{-}R$ be the left exact radical corresponding to $(\mathcal{T}_S, \mathcal{F}_S)$.

Let $\mathcal{I}_S$ be the ideal of $\mathrm{Mod}\text{-}R$ defined, for every A_R, B_R, by

$$\mathcal{I}_S(A_R, B_R) := \{\, f \in \mathrm{Hom}(A_R, B_R) \mid f(A_R) \subseteq t_S(B_R)\,\}.$$

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Equivalently, $\mathcal{I}_S(A_R, B_R)$ consists of all morphisms $f \in \mathrm{Hom}(A_R, B_R)$ that factor through a module in $\mathcal{T}_S$, so that $\mathrm{Mod}\text{-}R/\mathcal{I}_S$ is the stable category of $\mathrm{Mod}\text{-}R$ modulo the subcategory $\mathcal{T}_S$.

The case of $\mathrm{Mod}\text{-}R$

Theorem

Let R be a ring, $\mathcal{S}$ a set of representatives of the simple right R-modules up to isomorphism, and, for every $S \in \mathcal{S}$,

$$\mathcal{I}_S(A_R, B_R) = \{\, f \in \mathrm{Hom}(A_R, B_R) \mid f(A_R) \subseteq t_S(B_R) \,\}.$$

Then:
(1) *For every $S \in \mathcal{S}$, the category $\mathrm{Mod}\text{-}R/\mathcal{I}_S$ is subdirectly irreducible.*
(2) *The canonical functor $\mathrm{Mod}\text{-}R \to \prod_{S \in \mathcal{S}} \mathrm{Mod}\text{-}R/\mathcal{I}_S$ is a subdirect embedding.*